(4) A study of the necessary interval between current reversals in the process of demagnetisation has been made, and it is shown that the delay in reversal of magnetic phenomena in considerable masses of iron, due to eddy currents, is extremely small when the magnetic inductions are less than 300 C.G.S. units.

On the Dynamics of Revolving Fluids.

By Lord Rayleigh, O.M., F.R.S.

(Received December 8, 1916.)

So much of meteorology depends ultimately upon the dynamics of revolving fluid that it is desirable to formulate as clearly as possible such simple conclusions as are within our reach, in the hope that they may assist our judgment when an exact analysis seems impracticable. An important contribution to this subject is that recently published by Dr. Aitken.* It formed the starting point of part of the investigation which follows, but I ought perhaps to add that I do not share Dr. Aitken's views in all respects. His paper should be studied by all interested in these questions.

As regards the present contribution to the theory it may be well to premise that the limitation to symmetry round an axis is imposed throughout.

The motion of an inviscid fluid is governed by equations of which the first expressed by rectangular co-ordinates may be written

$$\frac{du'}{dt} + u'\frac{du'}{dx} + v'\frac{du'}{dy} + w'\frac{du'}{dz} = -\frac{dP}{dx}, \tag{1}$$

where

$$P = \int dp/\rho - V, \tag{2}$$

and V is the potential of extraneous forces. In (2) the density ρ is either a constant, as for an incompressible fluid, or at any rate a known function of the pressure p. Referred to cylindrical co-ordinates r, θ, z, with velocities u, v, w, reckoned respectively in the directions of r, θ, z increasing, these equations become†

$$\frac{du}{dt} + u\frac{du}{dr} + v\left(\frac{du}{rd\theta} - \frac{v}{r}\right) + w\frac{du}{dz} = -\frac{dP}{dr}, \tag{3}$$

* "The Dynamics of Cyclones and Anticyclones.—Part 3," 'Roy. Soc. Edin. Proc., vol. 36, p. 174 (1916).

† Compare Basset's 'Hydrodynamics,' § 19.

$$\frac{dv}{dt} + u\frac{dv}{dr} + v\left(\frac{dv}{rd\theta} + \frac{u}{r}\right) + w\frac{dv}{dz} = -\frac{d\mathrm{P}}{rd\theta}, \tag{4}$$

$$\frac{dw}{dt} + u\frac{dw}{dr} + v\frac{dw}{rd\theta} + w\frac{dw}{dz} = -\frac{d\mathrm{P}}{dz}. \tag{5}$$

For the present purpose we assume symmetry with respect to the axis of z, so that u, v, w, and P (assumed to be single-valued) are independent of θ. So simplified, the equations become

$$\frac{du}{dt} + u\frac{du}{dr} - \frac{v^2}{r} + w\frac{du}{dz} = -\frac{d\mathrm{P}}{dr}, \tag{6}$$

$$\frac{dv}{dt} + u\frac{dv}{dr} + \frac{uv}{r} + w\frac{dv}{dz} = 0, \tag{7}$$

$$\frac{dw}{dt} + u\frac{dw}{dr} + w\frac{dw}{dz} = -\frac{d\mathrm{P}}{dz}, \tag{8}$$

of which the second may be written

$$\left(\frac{d}{dt} + u\frac{d}{dr} + w\frac{d}{dz}\right)(rv) = 0, \tag{9}$$

signifying that (rv) may be considered to move with the fluid, in accordance with Kelvin's general theorem respecting "circulation." If r_0, v_0, be the initial values of r, v, for any particle of the fluid, the value of v at any future time when the particle is at a distance r from the axis is given by $rv = r_0v_0$.

Respecting the motion expressed by u, w, we see that it is the same as might take place with $v = 0$, that is when the whole motion is in planes passing through the axis, provided that we introduce a force along r equal to v^2/r. We have here the familiar idea of "centrifugal force," and the conclusion might have been arrived at immediately, at any rate in the case where there is no (u, w) motion.

It will be well to consider this case ($u = 0$, $w = 0$) more in detail. The third equation (8) shows that P is then independent of z, that is a function of r (and t) only. It follows from the first equation (6) that v also is a function of r only, and $\mathrm{P} = \int v^2 dr/r$. Accordingly by (2)

$$\int dp/\rho = \mathrm{V} + \int v^2 r^{-1} dr. \tag{10}$$

If V, the potential of impressed forces, is independent of z, so also will be p and ρ, but not otherwise. For example, if gravity (g) act parallel to z (measured downwards),

$$\int dp/\rho = \mathrm{C} + gz + \int v^2 dr/r, \tag{11}$$

gravity and centrifugal force contributing independently. In (11) ρ will be

constant if the fluid is incompressible. For gases following Boyle's law ($p = a^2\rho$),

$$a^2 (\log \rho, \text{ or } \log p) = C + gz + \int v^2 dr/r. \tag{12}$$

At a constant level the pressure diminishes as we pass inwards. But the corresponding rarefaction experienced by a compressible fluid does not cause such fluid to ascend. The heavier part outside is prevented from coming in below to take its place by the centrifugal force.*

The condition for equilibrium, taken by itself, still leaves v an arbitrary function of r, but it does not follow that the equilibrium is stable. In like manner an incompressible liquid of variable density is in equilibrium under gravity when arranged in horizontal strata of constant density, but stability requires that the density of the strata everywhere increase as we pass downwards. This analogy is, indeed, very helpful for our present purpose. As the fluid moves (u and w finite) in accordance with equations (6), (7), (8), (vr) remains constant (k) for a ring consisting always of the same matter, and $v^2/r = k^2/r^3$, so that the centrifugal force acting upon a *given portion* of the fluid is inversely as r^3, and thus a known function of position. The only difference between this case and that of an incompressible fluid of variable density, moving under extraneous forces derived from a potential, is that here the inertia concerned in the (u, w) motion is uniform, whereas in a variably dense fluid moving under gravity, or similar forces, the inertia and the weight are proportional. As regards the question of stability, the difference is immaterial, and we may conclude that the equilibrium of fluid revolving one way round in cylindrical layers and included between coaxial cylindrical walls is stable only under the condition that the circulation (k) always increases with r. In any portion where k is constant, so that the motion is there "irrotational," the equilibrium is neutral.

An important particular case is that of fluid moving between an inner cylinder ($r = a$) revolving with angular velocity ω and an outer fixed cylinder ($r = b$). In the absence of viscosity the rotation of the cylinder is without effect. But if the fluid were viscous, equilibrium would require†

$$k = vr = a^2\omega\,(b^2 - r^2)/(b^2 - a^2),$$

expressing that the circulation diminishes outwards. Accordingly a fluid without viscosity cannot stably move in this manner. On the other hand, if it be the outer cylinder that rotates while the inner is at rest,

$$k = vr = b^2\omega\,(r^2 - a^2)/(b^2 - a^2),$$

and the motion of an inviscid fluid according to this law would be stable.

* When the fluid is viscous the loss of circulation near the bottom of the containing vessel modifies this conclusion, as explained by James Thomson.

† Lamb's 'Hydrodynamics,' § 333.

We may also found our argument upon a direct consideration of the kinetic energy (T) of the motion. For T is proportional to $\int v^2 dr$, or $\int k^2 dr^2/r^2$. Suppose now that two rings of fluid, one with $k = k_1$ and $r = r_1$ and the other with $k = k_2$ and $r = r_2$, where $r_2 > r_1$, and of equal areas dr_1^2 or dr_2^2 are interchanged. The corresponding increment in T is represented by

$$(dr_1^2 = dr_2^2)\{k_2^2/r_1^2 + k_1^2/r_2^2 - k_1^2/r_1^2 - k_2^2/r_2^2\} = dr^2(k_2^2 - k_1^2)(r_1^{-2} - r_2^{-2}),$$

and is positive if $k_2^2 > k_1^2$; so that a circulation always increasing outwards makes T a minimum and thus ensures stability.

The conclusion above arrived at may appear to conflict with that of Kelvin,* who finds as the condition of minimum energy that the *vorticity*, proportional to $r^{-1}dk/dr$, must increase outwards. Suppose, for instance, that $k = r^{\frac{1}{2}}$, increasing outwards, while $r^{-1}dk/dr$ decreases. But it would seem that the variations contemplated differ. As an example, Kelvin gives for maximum energy

$$v = r \text{ from } r = 0 \text{ to } r = b,$$

$$v = b^2/r \text{ from } r = b \text{ to } r = a;$$

and for minimum energy

$$v = 0 \text{ from } r = 0 \text{ to } r = \sqrt{(a^2 - b^2)},$$

$$v = r - (a^2 - b^2)/r \text{ from } r = \sqrt{(a^2 - b^2)} \text{ to } r = a.$$

In the first case $$\int_0^a vr^2 dr = \tfrac{1}{4}b^2(2a^2 - b^2),$$

and in the second case $$\int_0^a vr^2 dr = \tfrac{1}{4}b^2;$$

so that the moment of momentum differs in the two cases. In fact Kelvin supposes operations upon the boundary which alter the moment of momentum. On the other hand, he maintains the strictly two-dimensional character of the admissible variations. In the problem that I have considered, symmetry round the axis is maintained and there can be no alteration in the moment of momentum, since the cylindrical walls are fixed. But the variations by which the passage from one two-dimensional condition to another may be effected are not themselves two-dimensional.

The above reasoning suffices to fix the criterion for stable equilibrium; but, of course, there can be no actual transition from a configuration of unstable equilibrium to that of permanent stable equilibrium without dissipative forces, any more than there could be in the case of a heterogeneous liquid under gravity. The difference is that in the latter case dissipative

* 'Nature.' vol. 23, October, 1880; 'Collected Papers,' vol. 4, p. 175.

forces exist in any real fluid, so that the fluid ultimately settles down into stable equilibrium, it may be after many oscillations. In the present problem ordinary viscosity does not meet the requirements, as it would interfere with the constancy of the circulation of given rings of fluid on which our reasoning depends. But for purely theoretical purposes there is no inconsistency in supposing the (u, w) motion resisted while the the v-motion is unresisted.

The next supposition to $u = 0$, $w=0$ in order of simplicity is that u is a function of r and t only, and that $w=0$, or at most a finite constant. It follows from (8) that P is independent of z, while (6) becomes

$$\frac{du}{dt} + u\frac{du}{dr} - \frac{v^2}{r} = -\frac{dP}{dr}, \tag{13}$$

determining the pressure. In the case of an incompressible fluid u as a function of r is determined by the equation of continuity $ur = C$, where C is a function of t only; and when u and the initial circumstances are known, v follows. As the motion is now two-dimensional, it may conveniently be expressed by means of the vorticity ζ, which moves with the fluid, and the stream-function ψ, connected with ζ by the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} = 2\zeta. \tag{14}$$

The solution, appropriate to our purpose, is

$$\psi = 2\int dr\, r^{-1}\int \zeta r\, dr + A\log r + B\theta, \tag{15}$$

where A and B are arbitrary constants of integration. Accordingly

$$u = \frac{d\psi}{rd\theta} = \frac{B}{r}, \qquad v = \frac{d\psi}{dr} = \frac{2}{r}\int \zeta r\, dr + \frac{A}{r}. \tag{16}$$

In general, A and B are functions of the time, and ζ is a function of the time as well as of r.

A simple particular case is when ζ is initially, and therefore permanently, uniform throughout the fluid. Then

$$v = \zeta r + Ar^{-1}. \tag{17}^*$$

Let us further suppose that initially the motion is one of pure rotation, as of a solid body, so that initially $A = 0$, and that then the outer wall closes in. If the outer radius be initially R_0 and at time t equal to R, then at time t

$$A = \zeta(R_0^2 - R^2), \tag{18}$$

* It may be remarked that (17) is still applicable under appropriate boundary conditions even when the fluid is viscous.

since vr remains unchanged for a given ring of the fluid; and correspondingly,

$$v = \zeta\{r + (R_0^2 - R^2)r^{-1}\}. \tag{19}$$

Thus, in addition to the motion as of a solid body, the fluid acquires that of a simple vortex of intensity increasing as R diminishes.

If at any stage the u motion ceases, (6) gives

$$dp/dr = \rho v^2/r, \tag{20}$$

and thus

$$p/\rho = \zeta^2\{\tfrac{1}{2}r^2 + 2(R_0^2 - R^2)\log r - \tfrac{1}{2}(R_0^2 - R^2)^2 r^{-2}\} + \text{const.} \tag{21}$$

Since, as a function of r, v^2 continually increases as R diminishes, the same is true for the difference of pressures at two given values of r, say r_1 and r_2, where $r_2 > r_1$. Hence, if the pressure be supposed constant at r_1, it must continually increase at r_2.

If the fluid be supposed to be contained between two coaxial cylindrical walls, both walls must move inwards together, and the process comes to an end when the inner wall reaches the axis. But we are not obliged to imagine an inner wall, or, indeed, any wall. The fluid passing inwards at $r = r_1$ may be supposed to be removed. And it remains true that, if it there pass at a constant pressure, the pressure at $r = r_2$ must continually increase. If this pressure has a limit, the inwards flow must cease.

It would be of interest to calculate some case in which the (u, w) motion is less simple, for instance, when fluid is removed at a point instead of uniformly along an axis, or inner cylindrical boundary. But this seems hardly practicable. The condition by which v is determined requires the expression of the motion of individual particles, as in the so-called Lagrangian method, and this usually presents great difficulties. We may however, formulate certain conclusions of a general character.

When the (u, w) motion is slow relatively to the v motion, a kind of "equilibrium theory" approximately meets the case, much as when the slow motion under gravity of a variably dense liquid retains as far as possible the horizontal stratification. Thus oil standing over water is drawn off by a syphon without much disturbing the water underneath. When the density varies continuously the situation is more delicate, but the tendency is for the syphon to draw from the horizontal stratum at which it opens. Or if the liquid escapes slowly through an aperture in the bottom of the containing vessel, only the lower strata are disturbed. In like manner when revolving fluid is drawn off in the neighbourhood of a point situated on the axis of rotation, there is a tendency for the surfaces of constant circulation to remain

cylindrical and the tendency is the more decided the greater the rapidity of rotation. The escaping liquid is drawn always from along the axis and not symmetrically in all directions, as when there is no rotation. The above is, in substance, the reasoning of Mr. Aitken, who has also described a simple experiment in illustration.

P.S.—It may have been observed that according to what has been said above the stability of fluid motion in cylindrical strata requires only that the *square* of the circulation increase outwards. If the circulation be in both directions, this disposition involves discontinuities, and the stability exists only under the condition that symmetry with respect to the axis is rigorously maintained. If this limitation be dispensed with, the motion is certainly unstable, and thus the stability of motion in cylindrical layers really requires that the circulation be one-signed. On the general question of the *two-dimensional* motion of liquids between fixed co-axial cylindrical walls reference may be made to a former paper.* The motion in cylindrical strata is stable provided that the "rotation either continually increase or continually decrease in passing outwards from the axis." The demonstration is on the same lines as there set out for plane strata.

* 'Proc. Lond. Math. Soc.,' vol. 11, p. 57 (1880); 'Scientific Papers,' vol. 1, p. 487. See last paragraph.